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# Technical Note

# Heat transfer in turbulent pipe flow revisited: similarity law for heat and momentum transport in low-Prandtl-number fluids

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**Nomenclature**

$D$ diameter [m]
$f$ Fanning friction factor
$Nu$ Nusselt number
$Pe$ Peclet number, $Re \cdot Pr$
$Pr$ Prandtl number
$r$ radial coordinate [m]
$Re$ Reynolds number
$T$ temperature [K]
$u$ axial velocity [m s$^{-1}$]
$u'$ fluctuating component of velocity [m s$^{-1}$]
$x$ distance from the wall [m].

*Greek symbols*

$\alpha$ thermal diffusivity [m$^2$ s$^{-1}$]
$\delta_M, \delta_H$ nondimensional momentum and heat penetration depths
$\nu$ momentum diffusivity [m$^2$ s$^{-1}$]
$\rho$ density [kg m$^{-3}$].

*Subscripts*

b fluid bulk
H uniform heat flux at the wall
n direction normal to the wall
T uniform wall temperature
w wall
$z$ axial direction.

## 1. Introduction

The task of studying transport phenomena in turbulent flow within a completely rational framework has, so far, been beset by current lack of understanding of turbulence. Nevertheless, the need for reliable tools to design heat and mass transfer equipment has prompted considerable research effort over several decades. That has led to a number of empirical models of turbulent transport.

Two approaches have been utilized to model the transport process at a wall boundary: one is based on the *Prandtl mixing length* concept and *eddy diffusivity*, the second one is based on the *surface renewal* concept first introduced by Danckwerts [1]. Eddy diffusivity models have proved quite useful in a large variety of cases. However, these empirical models provide little insight into the real mechanism underlying turbulent transport. As a result, the thermal eddy diffusivity, as related to the momentum eddy diffusivity, cannot be assessed a priori without further empirical assumptions.

More recently, the periodic surface renewal idea has received much attention in modeling heat and mass transfer in a turbulent flow. The basic assumption considers the surface to be covered by a mosaic of laminar flowing patches of fluid, where transport occurs only by molecular diffusion. These fluid patches are supposed to be periodically replaced by new ones (surface renewal model) [2] or to form viscous layers that periodically grow and collapse (growth-breakdown model) [3]. These are unsteady state one dimensional models. It has also been proposed that the fluid patches are arranged in a regular repetitive pattern of steady state boundary layers developing for a given length [4]. These simple models are able to explain a number of qualitative features observed in turbulent transport. In order to improve quantitative agreement with experimental data, these early concepts have been extensively elaborated.

It is not our intent to provide a critical review of the


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many eddy diffusivity or surface renewal models. However, it should be borne in mind that these models appeal to postulated mechanistic pictures of turbulence and that the concepts and the quantities involved have no fundamental relationship with the correlated turbulent fluctuations, the sole quantities that actually determine turbulent transport. The purpose of this work is to construct a theory able to predict turbulent heat transport from fundamental information, namely the thermal diffusivity and the normal turbulence intensity in the fluid bulk. The theory applies to heat transfer in turbulent incompressible flow for $Pr \ll 1$. The theoretical predictions are compared with available experimental data and empirical correlations to heat transport in liquid metals. For the sake of simplicity we shall refer to flow through cylindrical tubes in the following analysis. However, the results apply for arbitrarily shaped ducts as well.

## 2. Theory

### 2.1. Turbulent transport near a wall boundary

With reference to steady state incompressible flow through a circular pipe, two terms contribute to the radial momentum flux per unit area. The contribution due to molecular diffusion may be written as $\Phi_L(\hat{x}) = -(v/x)\rho u(x)$, $0 \leqslant \hat{x} \leqslant x$. The contribution deriving from the correlated turbulent fluctuations is $\Phi_T = \rho\overline{u'_z u'_n}(x)$. By virtue of $\overline{u'_z} = 0$ and $\overline{u'_n} = 0$ the following inequality subsists:

$$|\Phi_T(\hat{x})| \leqslant \rho\sqrt{\overline{u'^2_z}(\hat{x})} \cdot \sqrt{\overline{u'^2_n}(\hat{x})}.$$

$\sqrt{\overline{u'^2_z}}$ is a fraction of the local average velocity $u(x)$ [5]. $\sqrt{\overline{u'^2_n}}$ takes on finite values and is zero at the wall. As $x$ approaches zero, $v/x$ becomes much greater than $\sqrt{\overline{u'^2_n}}$, whereas $\sqrt{\overline{u'^2_z}}$ is always smaller than $u(x)$. The result is: $|\Phi_T| \ll |\Phi_L|$ near the wall. Thus, although turbulent fluctuations of the fluid velocity do take place even very close to the wall surface, turbulence does not contribute to momentum transfer within a region sufficiently close to the wall. This region will be subsequently termed *momentum diffusion region*. An analogous conclusion may be arrived at for heat transfer in turbulent flow. However, the diffusion regions for momentum and heat transfer are, in general, different. It follows that the momentum and energy equations, valid in the laminar regime, also apply to turbulent flow within the momentum and the heat diffusion regions respectively. The nondimensional velocity $u^* = u/u_b$ as a function of the nondimensional radial coordinate $r^* = r/D$ is readily obtained:

$$u^* = \left(\frac{du^*}{dr^*}\right)_{r^*=1/2}\left[r^{*2} - \frac{1}{4}\right] \tag{1a}$$

*also valid for turbulent flow*, within the momentum diffusion region. In the case of a uniformly heated slug flow, an analogous expression applies for the nondimensional temperature $T^* = (T - T_w)/(T_b - T_w)$:

$$T^* = \left(\frac{dT^*}{dr^*}\right)_{r^*=1/2}\left[r^{*2} - \frac{1}{4}\right] \tag{1b}$$

*The latter applies for turbulent flow as well*, within the heat diffusion region, provided that the Prandtl number is very small and the viscous heat source is negligible. For laminar and slug flow equations (1a) and (1b) apply over the entire tube radius. This enables $(du^*/dr^*)_{r^*=1/2}$ and $(dT^*/dr^*)_{r^*=1/2}$ to be calculated. However, in the turbulent regime eqns (1) are invalid far from the wall. In the latter case the velocity gradient essentially extinguishes within a short distance from the wall (herein referred to as *momentum penetration depth*), where the contribution from molecular diffusion to the total momentum flux cannot be disregarded. Apparently, the 'slope' of the nondimensional velocity profile: $(du^*/dr^*)_{r^*=1/2}$, prescribed by eqn (1a), is only dependent on the momentum penetration depth scaled to $D$. Likewise, in the turbulent regime, the scaled heat penetration depth dictates the 'slope' of the nondimensional temperature profile near the wall (eqn (1b)). Importantly, it appears that, in the turbulent regime, $(du^*/dr^*)_{r^*=1/2}$ and $(dT^*/dr^*)_{r^*=1/2}$ depend respectively on the scaled momentum and heat penetration depths through the same kind of relationship. This point is discussed in detail in the next paragraph. It is also shown in that paragraph how these scaled lengths are fundamentally related to certain nondimensional numbers, which account for turbulence. Here we observe that, since eqns (1a) and (1b) are formally identical, the dimensionless velocity and temperature profiles are also quantitatively identical whenever the scaled momentum and heat penetration depths are equal.

### 2.2. The penetration depth

In turbulent pipe flow, in addition to momentum flux, radial heat or mass flux may also occur by two mechanisms. One mechanism involves processes at a molecular scale, the other one involves intermittent *bulk* fluid motion, though at a local scale, in the radial direction. These two mechanisms act simultaneously. It was observed in the previous paragraph that the molecular mechanism prevails in the vicinity of the wall (diffusion region). In such region a velocity (temperature) gradient persists. Conversely, far from the wall, momentum (heat) is prevailingly carried by the radial fluctuating fluid motion. In this region, which is sometimes referred to as *turbulent core*, the time averaged fluid velocity (temperature) is comparatively uniform. There must also be a transition region where both mechanisms are important and the velocity (temperature) profile levels off. In order to estimate at which distance from the wall such an intermediate region is located, one should compare the transfer rate from the wall to the fluid due to molecular diffusion to the one due to turbulence. The rate of momentum (heat) diffusion from the wall to a surface in the fluid at a distance $x$ is: $\nu/x$ $(\alpha/x)$. On the other hand, the time averaged rate of momentum (heat) transfer through this surface due to the radial intermittent fluid motion is: $\sqrt{u_n'^2}(x)$. Comparison of these two rates affords an estimate of the distance of the transition region from the wall, i.e., the momentum penetration depth: $\hat{x}_M \sim \nu/\sqrt{u_n'^2}$. As a result the following scaling behavior subsists for the nondimensional momentum penetration depth:

$$\delta_M \propto \frac{\nu}{D\sqrt{u_n'^2}} \tag{2a}$$

An analogous expression applies for the heat penetration depth,

$$\delta_H \propto \frac{\alpha}{D\sqrt{u_n'^2}} \tag{2b}$$

$\sqrt{u_n'^2}$ should be evaluated at $x = \delta_M \cdot D$ and $x = \delta_H \cdot D$ respectively in expressions (2a) and (2b). In the turbulent core, $\sqrt{u_n'^2}$ is only slightly dependent on the wall distance [5, 6]. Moreover, experimental data by Laufer show that the dependency of the normal turbulence intensity scaled to the friction velocity $u_\tau$ on the Reynold number is fairly weak in the turbulent core [6]. It can be calculated using Laufer's data that $\sqrt{u_n'^2}/u_b$ is also weakly dependent on *Re* within this region.[1] Accordingly, we regard $\sqrt{u_n'^2}/u_b$ as a constant in the turbulent core. The error introduced by the latter approximation is unimportant as discussed in the next section. The turbulent core extends from the centerline of the pipe up to the momentum penetration depth. As a result, expressions (2a) and (2b) assume a more convenient form:

$$\delta_M \propto \frac{\nu}{Du_b} \tag{3a}$$

$$\delta_H \propto \frac{\alpha}{Du_b} \tag{3b}$$

It is noteworthy that, while relation (3a) is always valid, relation (3b) holds true for $Pr \ll 1$ only. In fact, relation (3b) requires that $\sqrt{u_n'^2}$—evaluated at $x = \delta_H \cdot D$—scale as $u_b$, which is the case only if $\delta_H > \delta_M$. The latter condition is satisfied when *Pr* is much less than unity. The right-hand sides of relations (3a) and (3b) are the reciprocal of the Reynolds and the Peclet number respectively.

It was anticipated in the previous paragraph that, given the formal identity of eqns (1a) and (1b) and of their respective boundary conditions, $(\mathrm{d}u^*/\mathrm{d}r^*)_{r^*=1/2}$ and $(\mathrm{d}T^*/\mathrm{d}r^*)_{r^*=1/2}$ depend on the scaled momentum and heat penetration depth, respectively, in the same way. This point is more thoroughly discussed hereafter. In fact, there is one subtle difference between the momentum and the heat transfer problem. In the very low Prandtl limit $\delta_M$ is much smaller than $\delta_H$. Accordingly, $\sqrt{u_n'^2}$ is nearly constant over essentially the entire heat penetration depth. On the other hand, $\sqrt{u_n'^2}$ is *not* constant within the momentum penetration depth and it becomes larger as the wall distance increases. We discuss below whether this difference implies different dependencies of $(\mathrm{d}u^*/\mathrm{d}r^*)_{r^*=1/2}$ and $(\mathrm{d}T^*/\mathrm{d}r^*)_{r^*=1/2}$ on $\delta_M$ and $\delta_H$ respectively. In this regard we observe that the velocity (temperature) profile in the neighborhood of $x = \delta_M \cdot D$ $(x = \delta_H \cdot D)$ depends solely on the ratio between $\nu/x$ $(\alpha/x)$ and $\sqrt{u_n'^2}(x)$ evaluated at $x = \delta_M \cdot D$ $(x = \delta_H \cdot D)$, independent of whether $\sqrt{u_n'^2}(x)$ is a constant or increases with the wall distance inside the penetration depth. On the other hand, if, for a chosen fluid, the momentum (heat) penetration depth and the velocity (temperature) profile in the *neighborhood* of $x = \delta_M \cdot D$ $(x = \delta_H \cdot D)$ are assigned—the latter condition being equivalent to assigning $\sqrt{u_n'^2}$ at $x = \delta_M \cdot D$ $(x = \delta_H \cdot D)$—it is plausible that the radial momentum (heat) flux at $x = \delta_M \cdot D$ $(x = \delta_H \cdot D)$ is determined. Accordingly, $(\mathrm{d}u/\mathrm{d}r)_{r=D/2}$ and $(\mathrm{d}T/\mathrm{d}r)_{r=D/2}$ would also be determined, whatever $\sqrt{u_n'^2}(x)$ *inside* the penetration depth. The above argument suggests that the dependency of $\sqrt{u_n'^2}$ on $x$ inside the penetration depth—which renders the momentum transfer problem different from the heat transfer problem—does not affect the velocity or temperature gradient at the wall. It would follow that $(\mathrm{d}u^*/\mathrm{d}r^*)_{r^*=1/2}$ and $(\mathrm{d}T^*/\mathrm{d}r^*)_{r^*=1/2}$ are functions of the sole scaled momentum and heat penetration depth respectively and that these functions are identical.

This result affords a fundamental analogy between heat and momentum transfer. Since the nondimensional velocity and temperature gradients at the wall relate to the Fanning friction factor $f$ and to the Nusselt number respectively as:

$$f \cdot \frac{Re}{2} = -\left(\frac{\mathrm{d}u^*}{\mathrm{d}r^*}\right)_{r^* 1/2}, \quad Nu = -\left(\frac{\mathrm{d}T^*}{\mathrm{d}r^*}\right)_{r^*=1/2},$$

the analogy may also be stated as follows:

(a) The Nusselt number is a function of the Peclet number only.
(b) The relationship that links *Nu* to *Pe* is identical with the one relating $f \cdot Re/2$ to *Re*.

The difference should be noted between the Reynolds analogy and the one derived here. This analogy applies for $Pr \ll 1$ and uniform heat flux at the wall surface. The case of constant wall temperature is investigated next.

### *2.3. Constant wall temperature*

The heat penetration depth is much smaller than the curvature radius of the wall, when the Peclet number is

[1] For instance, $\sqrt{u_n'^2}/u_\tau$ evaluated at $r^* = 0.45$ increases by only 11% as *Re* varies from $5 \times 10^4$ to $5 \times 10^5$, whilst $\sqrt{u_n'^2}/u_b$ decreases by 15% approximately.

sufficiently high. Accordingly, the curvature of the wall is disregarded in the following analysis and we shall refer to turbulent flow through a slit. However, the results are anticipated to be quite general, to a good approximation. At first let us consider slug flow through a slit. The expressions for the nondimensional temperature profiles with uniform heat flux and constant wall temperature read:

$$T^* = \left(\frac{dT^*}{dx^*}\right)_{x^*=0}\left[x^* - \frac{x^{*2}}{2}\right] \quad \text{(constant heat flux)} \tag{4a}$$

$$T^* = \sqrt{\left(\frac{dT^*}{dx^*}\right)_{x^*=0}} \cdot \sin\left(x^*\sqrt{\left(\frac{dT^*}{dx^*}\right)_{x^*=0}}\right) \quad \text{(constant wall temperature)} \tag{4b}$$

$x^*$ is the distance from the wall divided by the half-width of the slit. Heat conduction along the axis has been neglected in the derivation of eqn (4b). This approximation is acceptable if *Pe* is sufficiently high. The dimensionless temperature gradient at the wall is equal to 3 for uniformly heated flow or to $\pi^2/4$ for uniform wall temperature. Equations (4a) and (4b) apply for slug flow as well as in the heat diffusion region of a turbulent flow, provided that *Pr* is very small. Accordingly, the nondimensional temperature profiles for these two cases are geometrically akin. However, they extend over two different length scales: $0 \leqslant x^* \leqslant 1$ for slug flow; $0 \leqslant x^* \leqslant \delta_H$ for turbulent flow. As a result, the ratio: $(dT^*/dx^*)_{\substack{x^*=0\\ \text{turbulent}}}/(dT^*/dx^*)_{\substack{x^*=0\\ \text{slug}}}$ scales as the ratio between these two lengths: $1/\delta_H$. This holds true for uniform heat flux as well as for uniform temperature at the wall. This conclusion, along with knowledge of $(dT^*/dx^*)_{\substack{x^*=0\\ \text{slug}}}$ and expression (3b), affords a relationship, also *valid for turbulent flow*, between the Nusselt numbers at constant heat flux $Nu_H$ and constant wall temperature $Nu_T$ at the same Peclet number:

$$Nu_T = \frac{\pi^2}{12} Nu_H \tag{5}$$

Equation (5) applies to a turbulent incompressible flow for low *Pr* and high *Pe*.

## 3. Comparison with experiments and discussion

### 3.1. Constant heat flux

According to the analogy obtained in the previous section, Nusselt numbers have been calculated, at several *Pe*, using the friction factor chart for hydraulically smooth tubes. These are plotted in Fig. 1 together with experimental data points and empirical correlations for heat transfer in fluids with very low Prandtl number (liquid metals). It is well known that obtaining good heat transfer data for molten metals and metal alloys is a difficult task, also owing to uncertainty of the physical properties, ensuing from oxides formation. As a consequence, literature experimental data for liquid metals are somewhat scattered.

It should be noted that the present theory relies on the assumption, suggested by experimental evidence, that, in the turbulent core, the relative turbulence intensity normal to the wall, $\sqrt{\overline{u_n'^2}}/u_b$, is independent of the Reynolds number as well as of the wall distance. Actually, after an abrupt increase in the diffusion region, the radial turbulence intensity attains a weak maximum (peak) near the wall and decreases very slightly as the wall distance increases further [5, 6]. Moreover, the peak grows slightly and shifts towards the wall boundary at increasing *Re* [6]. This is reflected in a weak explicit dependency of $\delta_H$ and of *Nu* on the Reynolds number. However, this is a small effect, as also supported by recent empirical correlations for heat transfer to liquid metals [7, 8]. The latter predicts a dependency on $Re^{0.01}$. In fact, the results of the present theory agree with experimental data remarkably well (Fig. 1). This confirms that considering the radial turbulence intensity to be constant in the turbulent core is, in fact, a good approximation.

### 3.2. Constant wall temperature

Equation (5) shows that the ratio between the Nusselt moduli at constant wall temperature and constant heat flux is approximately $\pi^2/12$ ($\sim 0.82$) at low Prandtl number. This result is exact for $\delta_H \ll 1$, i.e., for high values of the Peclet number. An order of magnitude estimate of $\delta_H$ may be obtained from the Nusselt number: $\delta_H \sim 1/Nu$. So far as heat transfer to liquid metals is concerned, $\delta_H$ ranges from 0.01 to 0.1 in most cases of practical interest. Hence, eqn (5) is anticipated to be satisfactorily correct in such cases. Calculations predict that the ratio between the two moduli ranges from 0.73 to 0.88, depending upon the Peclet number [9]. Few accurate experimental studies are available on turbulent heat transfer in molten metals flowing through pipes at uniform wall temperature. Nusselt numbers at constant wall temperature have been obtained from data at constant heat flux using eqn (5) and are compared with experimental data and the empirical correlation to be found in the literature [10] in Fig. 2. Even in this case the agreement between theoretical predictions and experiments is remarkably good, especially at high *Pe*.

## 4. Concluding remarks

The analysis presented in this work afforded a new analogy between heat and momentum transport. The analogy applies to turbulent incompressible flow through an arbitrarily shaped duct for $Pr \ll 1$. The correlated

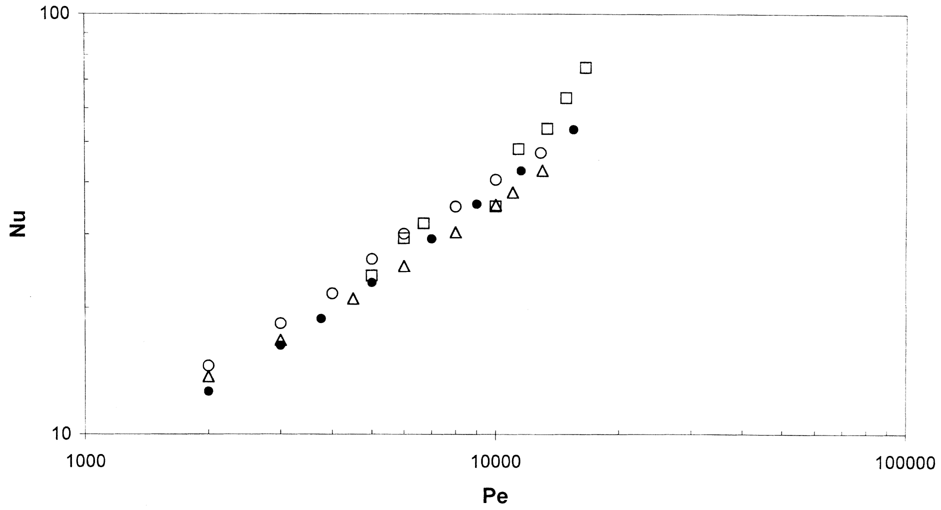


Fig. 1. *Nu* as a function of *Pe* for circular tubes with uniform heat flux at the wall: •—this study; ○—ref. [12]; □— ref.[11]; △—ref. [7].

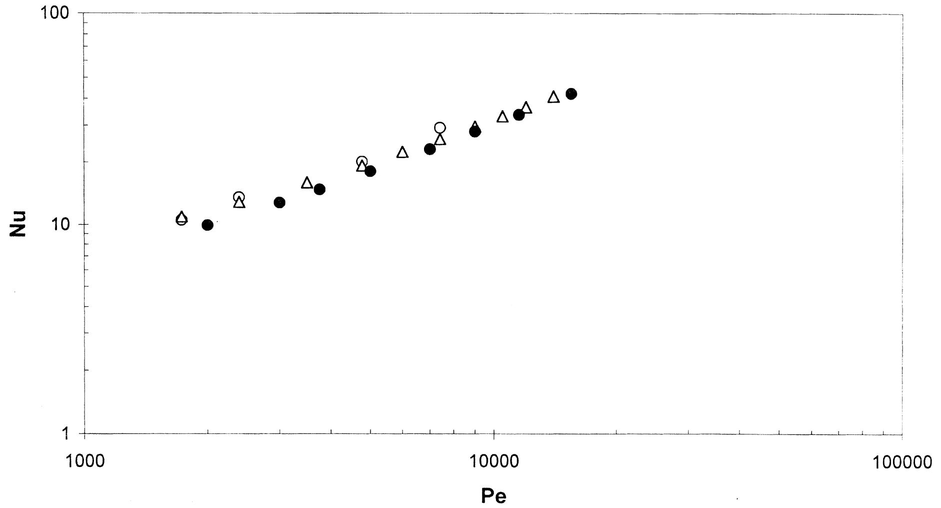


Fig. 2. *Nu* as a function of *Pe* for circular tubes with uniform wall temperature: •—this study; ○—ref. [10], experimental data points; △—ref. [10], empirical correlation.

turbulent fluctuations are the fundamental quantities underlying turbulent transport. Previous models have either proposed empirical expressions for such quantities (eddy diffusivity models) or postulated a physical picture of the transport mechanism at the wall surface (surface renewal). A rational approach to transport in turbulent flow must be based on the primary ingredients of turbulent transport, i.e., the turbulence fluctuation and the molecular momentum, heat or mass diffusivity. The present theory is derived directly from these first principles. This is the main feature of the present work. The roles of the velocity fluctuation normal to the wall and of the molecular diffusivity in turbulent transport are addressed. Also demonstrated, with scaling arguments, is how the interplay of these two quantities determines the momentum or heat transport rate. It is known that heat transfer data and correlations are generally not available for complex equipment with enhanced heat transfer capability. Obtaining heat transfer data for such devices is a difficult task, especially if the fluid is a liquid metal. From a practical perspective, the analogy derived in this work could be used to infer correlations for heat transfer in some systems with complex geometry, simply from pressure drop measurements.

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